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# Conformal invariance for polymers and percolation

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**Abstract.** We study some conformal invariance properties of the polymer and percolation problems in two dimensions. By analysing the transfer matrix spectrum of these models at criticality, we identify their series of thermal and magnetic exponents. Our results for percolation agree with the recent conjectures of Dotsenko and Fateev while some of our results for polymers are different. In the case of polymers, we interpret these series as a new set of geometrical exponents. In each case we also discuss the question of corrections to scaling.

## 1. Introduction

A statistical system at its critical point is supposed to be not only scale invariant but also conformally invariant. This property has some general consequences (valid for any dimension $D \geqslant 2$) which have been known for a long time (Polyakov 1970, Wegner 1976).

In the case $D = 2$ however, the conformal group, which is finite dimensional for $D > 2$, becomes infinite dimensional since it now consists of all the analytic transformations. Conformal invariance is thus a much stronger constraint here than in the general case and it has many important new implications which have received attention only recently (Belavin *et al* 1984).

The general structure of 2D conformal invariant field theories has been studied by Belavin *et al* (1984) and we refer the reader to the work of these authors for details. We shall simply recall here that the different theories are characterised by a single dimensionless number, the central charge $C$. In a given theory, the operators are classified into conformal blocks, each block consisting of one 'primary' and different 'secondary' operators. The properties of these secondary operators are simply related to the properties of the primary ones.

A discrete series of values of $C < 1$ has been singled out by Friedan *et al* (1984) who imposed the additional constraint of unitarity. For these values of $C$, there is a finite number of primary operators whose conformal block satisfies the unitarity requirement. Their dimensions are given by the Kac (1979) formula. The set formed by these operators and their descendants is stable with respect to operator product and can be considered as the operator algebra of a critical system. The associated physical systems have been identified as the critical and tricritical Ising model, the critical and tricritical three-state Potts model and the multicritical models of Andrews *et al* (1984) and Huse (1984). For these models the dimensions of all the different operators are thus known and the operator algebra and the correlation functions are determined (Belavin *et al* 1984, Dotsenko and Fateev 1984).

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However, there are many other models like the general O($n$) or $q$-state Potts model which are not present in the series of Friedan *et al* (1984). The conformal properties of these models are not completely known although some of them have been conjectured by Dotsenko and Fateev (1984). Of particular interest in this case are the polymer and percolation problems which correspond respectively to $n=0$ and $q=1$. Our purpose in this work is to study some properties of these two geometrical models by analysing the spectrum of their transfer matrix at the critical point. For doing this we shall use some recent results of Cardy (1984, 1985). Let us consider a primary operator (Belavin *et al* 1984) whose correlation function on the plane is given, using complex coordinates $z$, $\bar{z}$ by

$$\langle\Phi(z,\bar{z})\Phi(z',\bar{z})\rangle \simeq \frac{1}{(z-z')^{2h}}\frac{1}{(\bar{z}-\bar{z}')^{2\bar{h}}}. \tag{1}$$

In this formula, $h$ and $\bar{h}$ are related to the dimension and to the spin of the operator $\Phi$ by $d_\Phi = X_\Phi = h+\bar{h}$ and $s_\Phi = h-\bar{h}$. By using the logarithmic transformation which maps the plane onto a strip of width $l$ and periodic boundary conditions, Cardy (1984, 1985) has shown that it corresponds to the operator $\Phi$ an infinite number of eigenstates of the transfer matrix with eigenvalues $\lambda$ satisfying

$$-\log\frac{\lambda}{\Lambda} \simeq \frac{2\pi}{l}(X_\Phi + N + \bar{N}) \qquad (l\to\infty). \tag{2a}$$

In this formula $N$, $\bar{N}$ are arbitrary integers and $\Lambda$ is the largest eigenvalue of the transfer matrix. Since one works with periodic boundary conditions, the problem is invariant by translations perpendicular to the axis of the strip. The eigenvectors can thus be classified according to the value of their momentum $K$. As shown by Cardy (1984, 1985), the momentum corresponding to (2$a$) is

$$K = \frac{2\pi}{l}(s_\Phi + N - \bar{N}). \tag{2b}$$

The eigenstates at level $(N, \bar{N})$ ($N$ or $\bar{N} \neq 0$) correspond (Cardy 1985) to the secondary operators of the conformal block of $\Phi$ (Belavin *et al* 1984). These secondary operators are obtained by successive applications of the conformal generators $L_{-k}$, $\bar{L}_{-k'}$ and their general form is

$$L_{-k_1}\ldots L_{-k_m}\bar{L}_{-k'_1}\ldots\bar{L}_{-k'_{m'}}\Phi \tag{3}$$

where $k_1 \leqslant \ldots \leqslant k_m$, $k'_1 \leqslant \ldots \leqslant k'_{m'}$. If $N = \Sigma k_i$ and $\bar{N} = \Sigma k'_j$, the dimension of the operator (3) is $d_\Phi + N + \bar{N}$ and its spin is $s_\Phi + N - \bar{N}$, in agreement with (2).

By studying the transfer matrix spectrum one thus sees how one can obtain the dimensions of the different operators of the theory. In the following, we will mainly be interested in the determination of the thermal and magnetic series for the polymer and percolation problems.

In § 2 we consider the polymer (self-avoiding walk) problem. In this case the transfer matrix has a simple structure and we identify a rather large number of dimensions corresponding to two different series. Our results for the magnetic series agree with the conjecture of Dotsenko and Fateev (1984) while our results for the thermal series do not. From these dimensions we deduce a new infinite set of geometrical exponents. These exponents give for any value of $r$ the asymptotic behaviour of the number of $r$ polymers of length $\ell$ which are attached by their extremities.

In § 3 we study the percolation prolem. The transfer matrix in this case has a structure which is more complicated and we are just able to identify the first exponents of the thermal and magnetic series. These agree here with the conjecture of Dotsenko and Fateev (1984).

In each case we also discuss the problems of corrections to scaling.

## 2. The self-avoiding walk problem

### *2.1. General considerations*

It is well known that the problem of self-avoiding walks (polymers) on a regular lattice can be obtained by the analytic continuation to $n=0$ of the Heisenberg model with $n$ component classical spins (de Gennes 1972). If the Hamiltonian of the magnetic system is

$$\beta\mathcal{H} = -\beta \sum_{\langle ij \rangle} \mathbf{s}_i \cdot \mathbf{s}_j - H \sum_i s_i^1 \tag{4}$$

where $\mathbf{s}_i$ is a $n$-component vector with $\Sigma_{a=1}^{n} (s_i^a)^2 = n$, the partition function in the limit $n \to 0$ becomes (Sarma 1978, Gujrati 1981)

$$Z(\beta, H) = \left(1+\frac{H^2}{2}\right)^s \left(1+\sum_{\substack{r\geqslant 1 \\ \ell \geqslant r}} \Omega_{r\ell} H^{2r} \beta^{\ell} (1+\tfrac{1}{2}H^2)^{-r-\ell}\right). \tag{5}$$

In this formula $\Omega_{r\ell}$ is the number of configurations of $r$ non-intersecting self-avoiding walks of total length $\ell$ on the lattice of $S$ spins. In the limit $n \to 0$ the critical temperature of the model (4) becomes $\tau^c = (\beta^c)^{-1} = \mu$ where $\mu$ is the connectivity constant of the lattice.

We shall first study some properties of (5). In zero magnetic field all configurations with $r \geqslant 1$ polymers have a null weight so the partition function (5) becomes simply

$$Z(\beta, H=0) = 1 \tag{6a}$$

giving a free energy

$$f = (1/s) \log Z = 0. \tag{6b}$$

The result (6$b$) provides a simple way of determining the value of the central charge $C$ for the self-avoiding walk problem. It has indeed been shown recently by Blöte *et al* (1986) that the finite-size corrections to the free energy per site $f^{(l)}$ of a strip of width $l$ and periodic boundary corrections are given at the critical point by

$$f^{(l)} \simeq f + \pi C/6l^2 \tag{7}$$

where $f$ is the free energy in the thermodynamic limit. Since formula (6$b$) is also valid on strips and for any tempeature, we deduce from (7) that the value of $C$ in this case is $C=0$, a result which agrees with the formula of Dotsenko and Fateev (1984), giving $C(n)$ for the model (4) in the case $n=0$.

We note, however, that one can define a non-zero free energy for the self-avoiding walk problem by considering instead of $f$

$$f' = \lim_{n\to 0} \frac{f}{n} = \frac{\partial f}{\partial n}(n=0) \tag{8a}$$

which has a geometrical interpretation (Sarma 1978, Gujrati 1981)

$$f' = \sum_{\ell} \omega_{\ell}^{0} \beta^{\ell}. \tag{8b}$$

In this formula $\omega_{\ell}^{0}$ is the number of self-avoiding polygons of length $\ell$ per lattice site. The corresponding quantities $f'^{(l)}$ for strips of width $l$ then converge at the critical point $\beta = \beta^{c}$ as

$$f'^{(l)} \simeq f' + \frac{\pi}{6l^2} \frac{\partial C}{\partial n}(n=0) \tag{9}$$

and the corrections to scaling are determined in this case by $C' = \partial C/\partial n(n=0)$. We have calculated the $f'^{(l)}$ using the transfer matrix technique already introduced in Enting (1980), Klein (1980) and Derrida (1981). The lattice is the square lattice for which the critical point is determined with a good precision $\beta^{c} = \mu_{SQ}^{-1} = 0.379\,0528 \pm 0.000\,0025$ (Enting and Guttman 1985). We have done all our calculations with this estimate of $\beta^{c}$ and our results are given with an accuracy which is not sensitive to the uncertainty of $\beta^{c}$. (It is also possible to work on the hexagonal lattice for which $\beta^{c}$ is known exactly (Nienhuis 1982a, b). The results in this case are very similar to those obtained with the square lattice but the strip widths $l$ are more limited so we do not present these results here.) The same remarks apply to all the calculations of this paper. The $f'^{(l)}$ and successive estimates of $C'$ obtained by comparing $f'^{(l)}$ and $f'^{(l+1)}$ are given in table 1 for different values of $l$. As can be seen, these estimates converge rapidly to the value

$$C' = \frac{\partial C}{\partial n}(n=0) = \frac{5}{3\pi} = 0.530\,516 \tag{10}$$

obtained with the result of Dotsenko and Fateev (1984), thus confirming the conjecture of these authors. Values of $C(n)$ have also been calculated by Blöte *et al* (1986) in the case of the $n$-component cubic model.

**Table 1.** In this table we give the values of $f'^{(l)}$ (8*b*) at the critical point $\beta^{c} = \mu_{SQ}^{-1}$ and the estimates of $C' = \partial C/\partial n(n=0)$ obtained by comparing $f'^{(l)}$ and $f'^{(l+1)}$ using (9). The expected value is the result of Dotsenko and Fateev (1984).

| $l$ | $f'^{(l)}$ | $C' = \dfrac{\partial C}{\partial n}(n=0)$ |
|---|---|---|
| 2 | 0.120 06 | 0.678 92 |
| 3 | 0.070 68 | 0.627 12 |
| 4 | 0.054 72 | 0.588 24 |
| 5 | 0.047 79 | 0.566 17 |
| 6 | 0.044 17 | 0.554 12 |
| 7 | 0.042 03 | 0.547 26 |
| 8 | 0.040 66 | 0.543 06 |
| 9 | 0.039 73 | 0.540 31 |
| 10 | 0.039 06 | 0.538 39 |
| 11 | 0.038 57 | |
| Expected value | | $\dfrac{5}{3\pi} = 0.530\,516$ |

When $H$ is non-zero, the partition function (5) can be calculated with the transfer matrix technique recently introduced in Saleur and Derrida (1986). The method is a simple generalisation of what has already been done by Enting (1980), Klein (1980) and Derrida (1981) with the main difference that there are now several polymers on the strip. In this case the possible configurations at one column of the strip are defined by the pairs of sites which are related together by the left part of the strip and by all the sites which are extremities of polymers whose other extremity is in the left part of the strip.

The matrix elements between these configurations depend now on $\beta$ and $H$. If $H=0$, the largest eigenvalue of this transfer matrix is $\Lambda=1$, which corresponds to (6*a*) while the other eigenvalues describe the fall-off of different correlations. Since one cannot go from one column to the next by adding extremity sites which would imply some factors $H$, the transfer matrix takes a simple 'blockwise triangular' structure. For determining the complete spectrum, it suffices then to diagonalise submatrices† $M_r$ acting on configurations with $r$ extremity sites, which corresponds to $r$ non-intersecting very long polymers present on the strip. In § 2 we study the largest eigenvalue of the matrices $M_r$ while we give some details for the complete spectrum of $M_2$ in § 3.

### 2.2. *Study of the largest eigenvalue of the matrices $M_r$*

The transfer matrix $M_1$ has already been considered in Derrida (1981) and Saleur and Derrida (1986). In the limit $n\to 0$, the spin-spin correlation of the model (4) can be written

$$\lim_{n\to 0}\frac{1}{n}\langle \mathbf{s}_i\cdot\mathbf{s}_j\rangle=\sum_{\mathscr{G}_1}\beta^{\ell(\mathscr{G}_1)} \tag{11}$$

where the $\mathscr{G}_1$ are graphs formed by a self-avoiding walk of length $\ell$ with extremities in $i$ and $j$.

The largest eigenvalue $\Lambda_1$ of the transfer matrix with one polymer $M_1$ thus gives the fall-off of the spin-spin correlation on the strip. At the critical point $\beta^c$ it is related to the dimension $X_{H_1}$ of the spin operator by

$$-\frac{l}{2\pi}\log\Lambda_1^{(l)}\simeq X_{H_1}=\frac{\eta}{2}\qquad (l\to\infty). \tag{12}$$

The value of this dimension has already been found by Saleur and Derrida (1986) to be in good agreement with the conjectured value (Nienhuis 1982a) $X_{H_1}=\frac{5}{48}$. This can also be checked in the first column of table 2. In a similar way, because of formula (8*b*), the largest eigenvalue of the transfer matrix with two polymers $M_2$ gives the dimension of the energy operator

$$-\frac{l}{2\pi}\log\Lambda_2^{(l)}\simeq X_{T_1}=2-\frac{1}{\nu}\qquad (l\to\infty). \tag{13}$$

As can be seen in the first column of table 4, the results obtained in this way are in agreement with the conjectured value (Nienhuis 1982a) $X_{T_1}=\frac{2}{3}$.

† The matrices $M_r$ depend naturally on the width $l$ but for clarity we do not indicate it explicitly.

**Table 2.** Dimensions $x_{2l-1}$ obtained with the largest eigenvalues of the matrices $M_{2l-1}$ using formula (19).

| $l$ | $x_1^{(l)}$ | $x_3^{(l)}$ | $x_5^{(l)}$ | $x_7^{(l)}$ | $x_9^{(l)}$ |
|---|---|---|---|---|---|
| 1 | 0.1544 | | | | |
| 2 | 0.1292 | | | | |
| 3 | 0.1165 | 1.389 | | | |
| 4 | 0.1108 | 1.493 | | | |
| 5 | 0.1081 | 1.505 | 3.860 | | |
| 6 | 0.1068 | 1.565 | 4.093 | | |
| 7 | 0.1060 | 1.578 | 4.236 | 7.565 | |
| 8 | 0.1056 | 1.586 | 4.330 | 7.928 | |
| 9 | 0.1053 | 1.592 | 4.394 | 8.181 | 12.506 |
| 10 | 0.1050 | 1.595 | 4.439 | 8.365 | 12.997 |
| 11 | | | 4.472 | 8.502 | 13.368 |
| 12 | | | | 8.607 | 13.654 |
| 13 | | | | | 13.878 |
| 14 | | | | | 14.057 |

**Table 3.** Successive '*a posteriori*' estimates obtained from table 2. The conjectured values correspond to the magnetic series of Dotsenko and Fateev (1984) (see formula (18)).

| $l$ | $x_1^{(l)}$ | $x_3^{(l)}$ | $x_5^{(l)}$ | $x_7^{(l)}$ | $x_9^{(l)}$ |
|---|---|---|---|---|---|
| 2 | 0.0086 | | | | |
| 3 | 0.0978 | | | | |
| 4 | 0.1034 | 1.619 | | | |
| 5 | 0.1043 | 1.611 | | | |
| 6 | 0.1044 | 1.609 | 4.664 | | |
| 7 | 0.1044 | 1.608 | 4.642 | | |
| 8 | 0.1043 | 1.607 | 4.630 | 9.236 | |
| 9 | 0.1042 | 1.606 | 4.623 | 9.202 | |
| 10 | | | 4.619 | 9.177 | 15.335 |
| 11 | | | | 9.160 | 15.291 |
| 12 | | | | | 15.252 |
| 13 | | | | | 15.226 |
| Identified dimensions | $X_{H_1}=\frac{5}{48}$ $=0.1042$ | $X_{H_2}=\frac{77}{48}$ $=1.6042$ | $X_{H_3}=\frac{221}{48}$ $=4.6042$ | $X_{H_4}=\frac{437}{48}$ $=9.1042$ | $X_{H_5}=\frac{725}{48}$ $=15.1042$ |

Formulae (12) and (13) have a simple generalisation. Let us take, for example, the matrix $M_3$ and define the operator $\psi_i$ which is a local product of an energy and a spin:

$$\psi_i = s_{i_1} \cdot (s_{i_2} \cdot s_{i_3}) + \text{permutations} \tag{14}$$

where $i_1$, $i_2$, $i_3$ are some fixed points in the neighbourhood of $i$. We consider now the correlation function

$$\langle \psi_i \cdot \psi_j \rangle = \langle [s_{i_1} \cdot (s_{i_2} \cdot s_{i_3})] \cdot [s_{j_1} \cdot (s_{j_2} \cdot s_{j_3})] \rangle + \text{permutations}. \tag{15}$$

This is a sum of terms like

$$\langle s_{i_1}^a s_{i_2}^b s_{i_3}^b s_{j_1}^c s_{j_2}^d s_{j_3}^d \rangle \tag{16}$$

**Table 4.** Dimensions $x_{2,l}$ obtained with the largest eigenvalue of the matrices $M_{2,l}$ using formula (21).

| $l$ | $x_2^{(l)}$ | $x_4^{(l)}$ | $x_6^{(l)}$ | $x_8^{(l)}$ | $x_{10}^{(l)}$ |
|---|---|---|---|---|---|
| 2 | 0.6176 | | | | |
| 3 | 0.6570 | | | | |
| 4 | 0.6664 | 2.470 | | | |
| 5 | 0.6689 | 2.639 | | | |
| 6 | 0.6698 | 2.732 | 5.582 | | |
| 7 | 0.6695 | 2.787 | 5.856 | | |
| 8 | 0.6693 | 2.822 | 6.053 | 9.881 | |
| 9 | 0.6691 | 2.846 | 6.190 | 10.308 | |
| 10 | 0.6689 | 2.862 | 6.289 | 10.620 | 15.440 |
| 11 | | 2.874 | 6.361 | 10.854 | 15.995 |
| 12 | | | 6.416 | 11.033 | 16.426 |
| 13 | | | | 11.173 | 16.766 |
| 14 | | | | | 17.038 |

where $abcd \in [1, n]$. In the limit $n \to 0$, a term like (16) is calculated (Sarma 1978, Gujrati 1981) by contracting pairs of spins with the same component index and by drawing non-intersecting polymers relating the points whose spins have been contracted, with a factor $\beta$ for each polymer link. The different types of graphs are also weighted by different powers of $n$. This finally gives a formula similar to (11):

$$\lim_{n\to 0} \frac{1}{n} \langle \psi_i \cdot \psi_j \rangle = 4 \sum_{\mathcal{G}_1'} \beta^{\ell(\mathcal{G}_1')} + 6 \sum_{\mathcal{G}_3} \beta^{l(\mathcal{G}_3)} \tag{17}$$

where the graphs $\mathcal{G}_1'$ consist of a polymer relating one $i$ to one $j$, the other $i$ being related by a second polymer and the other $j$ by a third one while the graphs $\mathcal{G}_3$ consist of three polymers, each relating one $i$ to one $j$. Because of (11), the first term of (17) corresponds to spin-spin correlations. If one writes the short distance expansion of the product $\varepsilon s \sim s + \ldots$ one can then interpret the second term of (17) as describing the correlations of a new operator generated in the product $\varepsilon s$. The largest eigenvalue of $M_3$ gives then the dimension of this operator by a formula similar to (12) and (13). An argument of the same type applies to the other matrices $M_p$.

Dotsenko and Fateev (1984) have conjectured that the magnetic operators which are generated in the product of the spin with several energies belong to the conformal blocks of $\Phi_{3/2,t+1/2}$. (We denote by $\Phi_{p,q}$ the primary operator in the degenerate representation indexed by $p$ and $q$. We follow Dotsenko and Fateev (1984) by introducing half-integer indices, although their meaning is not yet clear.) The dimension of $\Phi_{3/2,t+1/2}$ is given by the Kac formula (1979) with $C = 0$

$$X_{H_t} = 2h_{3/2,t+1/2} = \frac{9(2t-1)^2 - 4}{48}. \tag{18}$$

Like the dimension $X_{H_1}$ which was obtained by the transfer matrix with one polymer $M_1$, we have obtained the dimensions $X_{H_t}$ by considering the largest eigenvalue of the matrices $M_{2t-1}$ with $2t-1$ polymers. In table 2 we give the values of

$$x_{2t-1}^{(l)} = -\frac{l}{2\pi} \log \Lambda_{2t-1}^{(l)} \tag{19}$$

for different widths $l$ up to $t=5$. In table 3 we give the corresponding '*a posteriori* estimates' which are obtained by assuming a pure power law convergence of the $x_{2t-1}^{(l)}$ for three different sizes and then extrapolating to $l=\infty$ (Derrida and Stauffer 1985). As can be seen from these tables, the results converge rapidly to the values $x_{2t-1}=X_{H_t}$ of (18), thus confirming the conjecture of Dotsenko and Fateev (1984) for the magnetic series†.

In the same way, Dotsenko and Fateev (1984) have also conjectured that the thermal operators which are generated in the product of several energies belong to the blocks of $\Phi_{2t+1,1}$. The corresponding dimensions are

$$2h_{2t+1,1}=\frac{(4t-1)^2-1}{12}. \tag{20}$$

We have already seen in (13) that the largest eigenvalue of $M_2$ gives the dimension of the energy operator, in agreement with $X_{T_1}=2h_{3,1}=\frac{2}{3}$. However, the other dimensions which are deduced from the largest eigenvalue of the matrices $M_{2t}$ by

$$x_{2t}^{(l)}=-\frac{l}{2\pi}\log\Lambda_{2t}^{(l)} \tag{21}$$

do not agree with (20), as can be seen in tables 4 and 5. On the contrary, these dimensions are in good agreement with another line of the conformal grid (see table 6) corresponding to

$$X_{T_t}=x_{2t}=2h_{3,t+2}. \tag{22}$$

This numerical study thus confirms the magnetic series of Dotsenko and Fateev (1984) but gives a thermal series which disagrees with the conjecture of these authors. It is possible however that the values (20) are present in the rest of the spectrum of the matrices $M_{2t}$. We shall thus give in the following some details of the spectrum of $M_2$.

**Table 5.** *A posteriori* estimates corresponding to table 4. The identified dimensions are $2h_{3,t+2}$ of table 6 (see the text and formula (22)).

| $l$ | $x_2^{(l)}$ | $x_4^{(l)}$ | $x_6^{(l)}$ | $x_8^{(l)}$ | $x_{10}^{(l)}$ |
|---|---|---|---|---|---|
| 3 | 0.6698 | | | | |
| 4 | 0.6706 | | | | |
| 5 | 0.6698 | 2.904 | | | |
| 6 | — | 2.936 | | | |
| 7 | 0.6698 | 2.929 | 6.760 | | |
| 8 | 0.6682 | 2.925 | 6.731 | | |
| 9 | 0.6675 | 2.923 | 6.712 | 12.095 | |
| 10 | | 2.922 | 6.701 | 12.056 | |
| 11 | | | 6.692 | 12.023 | 18.957 |
| 12 | | | | 12.001 | 18.908 |
| 13 | | | | | 18.864 |
| Identified dimensions | $X_{T_1}=\frac{2}{3}$ $=0.6667$ | $X_{T_2}=\frac{35}{12}$ $=2.9167$ | $X_{T_3}=\frac{20}{3}$ $=6.6667$ | $X_{T_4}=\frac{143}{12}$ $=11.9167$ | $X_{T_5}=\frac{56}{3}$ $=18.6667$ |

† This can be seen more precisely by using a graphical extrapolation of the data in table 2, as has already been done by Saleur and Derrida (1986) in the particular case of $X_{H_1}$. For brievity, we do not present such extrapolations here, the *a posteriori* estimates for the largest widths already being very close to the limiting values.

**Table 6.** Values of the conformal dimensions $h_{pq}$ obtained by the Kac formula in the case $C=0$: $h_{p,q}=[(2p-3q)^2-1]/24$. Note that certain authors (Friedan *et al* 1985) use $h_{q,p}$ for what is denoted here by $h_{p,q}$.

| $p$ \ $q$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 5/8 | 2 | 33/8 | 7 | 85/8 | 15 |
| 2 | 0 | 1/8 | 1 | 21/8 | 5 | 65/8 | 12 |
| 3 | 1/3 | −1/24 | 1/3 | 35/24 | 10/3 | 143/24 | 28/3 |
| 4 | 1 | 1/8 | | $h_{p+3,q+2}=h_{p,q}$ | | | |
| 5 | 2 | 5/8 | | | | | |

### *2.3. Spectrum of the transfer matrix $M_2$*

We now present some details concerning the spectrum of $M_2$. We have already mentioned that the problem is left invariant by translations perpendicular to the axis of the strip. In fact, the problem is also invariant by symmetry with respect to this axis. The eigenspaces can thus be classified by $|K|$ only and the eigenvalues for $|K|\neq 0$ are twice degenerated (this corresponds to the exchange of $h\bar{h}$ or $N\bar{N}$ in formulae (1)-(3)). In tables 7-9 we study the first few dimensions deduced from the spectrum of $M_2$ at $|K|=0,\ 2\pi/l,\ 4\pi/l$. (For brevity we have simply given the last *a posteriori* estimates.) We have already identified the first dimension of table 7 as $X_{T_1}$, the dimension of the energy $\varepsilon$. In tables 7-9, we also observe this value shifted by different integers, corresponding to the other operators of the energy block (3). The second dimension of table 7 is identified with $X_{T_1}+2$, the first and third dimensions of table 8 with $X_{T_1}+1$ and $X_{T_1}+3$ and the second dimension of table 9 with $X_{T_1}+2$.

**Table 7.** *A posteriori* estimates of the first few dimensions deduced from the spectrum of $M_2$ at $K=0$.

| $l$ | | | | |
|---|---|---|---|---|
| 7 | 0.6698 | 2.704 | 4.064 | 5.781 |
| 8 | 0.6682 | 2.689 | 4.025 | 5.244 |
| 9 | 0.6675 | 2.681 | 4.014 | 4.906 |
| Identification of the dimensions | $X_{T_1}=0.6667$ | $X_{T_1}+2=2.6667$ | 4 | $d+1$ |

**Table 8.** As table 7 for $|K|=2\pi/l$.

| $l$ | | | | |
|---|---|---|---|---|
| 5 | 1.691 | 3.020 | 3.553 | 4.303 |
| 6 | 1.693 | 3.016 | 3.595 | 4.071 |
| 7 | 1.690 | 3.014 | 3.634 | 3.826 |
| Identification of the dimensions | $X_{T_1}+1=1.6667$ | 3 | $X_{T_1}+3=3.6667$ | $d\sim 3.5$ |

**Table 9.** As table 7 for $|K| = 4\pi/l$.

| $l$ | | | | |
|---|---|---|---|---|
| 5 | 1.967 | 3.017 | 3.703 | 5.118 |
| 6 | 2.067 | 2.843 | 3.871 | 4.885 |
| 7 | 2.053 | 2.759 | 3.920 | 4.7785 |
| Identification of the dimensions | 2 | $X_{T_1}+2=2.6667$ | 4 | $d+1$ |

In tables 7-9 are also present integer dimensions corresponding to some members of the identity block. The smallest one is the first dimension of table 9 which we identify with $d=2$. The associated operators are $L_{-2}I \sim T$ and $\bar{L}_{-2}I \sim \bar{T}$ where $T$ is the stress-energy tensor (Cardy 1985). We also observe this value shifted by integers like $d=3$ in table 8 or $d=4$ in tables 7 and 9 ($d=1$ is not observed since $L_{-1}I = \bar{L}_{-1}I = 0$).

Beside the dimensions corresponding to the energy and the identity blocks, we have also observed many other dimensions in the spectrum of $M_2$. For example the fourth column of table 8 corresponds to a new operator of spin $s=1$ and dimension $d \simeq 3.50$. In tables 7 and 9 one can also observe this value shifted by 1. It is difficult to extrapolate the results obtained with eigenvalues which are too far in the spectrum since the estimates converge more and more slowly and we have not been able to classify the new dimensions in series analogous to (18) or (22). We note simply that the complete spectrum seems to be very rich.

A curious feature is that the operator $\Phi_{5,1}$ which was conjectured by Dotsenko and Fateev (1984) to be generated in the product of two energies is not present in our spectrum. We have already mentioned that the dimension $2h_{5,1}=4$ is not observed by looking at the largest eigenvalue of the matrices $M_{2l}$. In a similar way, this dimension is not present in the rest of the spectrum of $M_2$ and we have reached the same conclusion for the other $M_r$. We have indeed observed the value $d=4$ in tables 7-9, but we have obtained it only once and we have shown that it corresponds to an operator of the identity bock, namely $L_{-2}\bar{L}_{-2}I$ which is present in any case (Cardy 1985). If $\Phi_{5,1}$ was present we should observe $d=4$ twice, as will be the case for percolation (see § 3). It should be interesting to understand whether $\Phi_{5,1}$ is indeed not present in the operator algebra of the self-avoiding walk problem or whether our transfer matrix does not contain all the operators.

### *2.4. Corrections to scaling for the self-avoiding walk problem*

We can now discuss the question of corrections to scaling in the self-avoiding walk problem. It is well known (Wegner 1972) that the presence of irrelevant thermal operators gives corrections to the power law behaviour of the different quantities at the critical point. For example the singular part of the free energy (8) near $\beta = \beta^c$ is

$$f_s \sim (\beta - \beta^c)^{2-\alpha} + \text{cste}(\beta - \beta^c)^{2-\alpha+\Delta} \dots \tag{23}$$

where $\Delta$ is the first corrections to scaling exponent. $\Delta$ is related to the dimension $X$ of the corresponding irrelevant operator by $\Delta = -Y\nu = (X-2)\nu$ ($-Y$ is also denoted by $\omega$ in the literature). We note here that non-linearities in the scaling fields can also

produce corrections to scaling (Aharony and Fisher 1983). This gives additional corrections to (23), the most singular being a term $(\beta-\beta^c)^{2-\alpha+1}$.

The determination of $\Delta$ by a direct analysis of numerical data is a difficult problem (Privman and Fisher 1983). For example in the case of the three-state Potts model, the analysis of different authors give a value $\Delta = 0.57 \pm 0.13$ (Adler and Privman 1982 and references therein) while the conformal invariance theory, which provides a complete picture of this model at criticality (Dotsenko 1984), gives $\Delta = \frac{2}{3} = 0.667$, a value which is rather different from this estimate, although it is compatible with the error bars.

For models which are not completely solved by the conformal invariance theory (like polymers on percolation), it is possible to get information on the dimensions of the different operators by studying the transfer matrix spectrum (see formula (2)). This should be an easier way to determine $\Delta$ than analysing confluent singularities in a series expansion. For the polymer problem we have found two different irrelevant thermal operators whose dimensions correspond to values of $\Delta$ smaller than one. These dimensions are $X_{T_1}+2=\frac{8}{3}$, giving $\Delta'=\frac{1}{2}=0.5$ and $X_{T_2}=\frac{35}{12}$ giving $\Delta=\frac{11}{16}=0.687$. However the dimension $X_{T_1}+2$ corresponds to the operator $L_{-1}\bar{L}_{-1}\varepsilon \sim \nabla^2\varepsilon$ which is a total derivative and cannot contribute to corrections to scaling (Cardy 1986). We are thus left with a single candidate $\Delta=\frac{11}{16}$.

The numerical estimates of the literature range from $\Delta \simeq 0.5$ to $\Delta \simeq 1$ (Adler 1983, Djordjevic *et al* 1983, Privman 1984, Rapaport 1985, Enting and Guttman 1985). This is in reasonable agreement with this value.

### *2.5. Geometrical interpretation of the thermal and magnetic series for the self-avoiding walk problem*

It is well known that the first magnetic exponent $X_{H_1}=\eta/2$ has a simple geometrical meaning.

We have already mentioned in formula (11) that the spin-spin correlation $\langle s_i \cdot s_j \rangle$ is related by a Laplace transform to the number of self-avoiding walks with extremities in $i$ and $j$, $\omega_{1\ell}(|i-j|)$. From the asymptotic form of this correlation function

$$\langle s_i \cdot s_j \rangle \sim |i-j|^{-\eta} E\left(\frac{|i-j|}{(\beta-\beta^c)^{-\nu}}\right) \tag{24a}$$

one can obtain (Sarma 1978, Gujrati 1981) by inverting (11)

$$\omega_{1\ell}(|i-j|) \sim \mu^{\ell}\ell^{-\eta\nu-1} F\left(\frac{|i-j|}{\ell^{\nu}}\right) \tag{24b}$$

(with $\mu^{-1}=\beta^c$). Integration then gives the number of self-avoiding walks of length $\ell$ per lattice site

$$\omega_{1\ell} \sim \mu^{\ell}\ell^{(2-\eta)\nu-1} = \mu^{l}\ell^{\gamma-1} \tag{24c}$$

where we have used the scaling relation $\gamma/\nu = 2-\eta$.

In the same way the study of energy-energy correlations gives the asymptotic form of the number of self-avoiding polygons of length $\ell'$ per lattice site (Sarma 1978, Gujrati 1981)

$$\omega^0_{\ell'} \sim \mu^{\ell'}\ell'^{-2\nu-1} = \mu^{\ell'}\ell'^{\alpha-3} \tag{25a}$$

where we have used the hyperscaling relation $2-\alpha=2\nu$. From (25*a*) we deduce the number of configurations of two polymers each of length $\ell$ which are attached by their extremities

$$\omega_{2\ell} \sim \mu^{2\ell} \ell^{\alpha-2}. \tag{25b}$$

These formula have a simple generalisation. Let us consider the correlation function defined by

$$\langle \Phi_i \Phi_j \rangle = \sum_{\mathcal{G}_r} \beta^{\ell(\mathcal{G}_r)} \tag{26}$$

where the $\mathcal{G}_r$ are graphs formed by $r$ non-intersecting self-avoiding walks relating a fixed neighbourhood of $i$ to a fixed neighbourhood of $j$. This correlation function calculated on strips has an exponential decay $\exp(-|i-j|/\xi)$ with $\xi^{(l)} = -(\log \Lambda_r^{(l)})^{-1}$ and $-(l/2\pi)\log \Lambda_r^{(l)} \simeq x_r$ (see (19) and (21)). The correlation function (26) thus decays on the plane like $|i-j|^{-2x_r}$. By an argument similar to (24) we deduce then that the number of configurations of $r$ non-intersecting self-avoiding walks of total length $\ell'$ which relate fixed neighbourhoods of two different points $i$ and $j$ behaves like

$$\omega_{r\ell'}(|i-j|) \sim \mu^{\ell'} \ell'^{-2x_r\nu-1} F\left(\frac{|i-j|}{\ell'^{\nu}}\right). \tag{27a}$$

If we fix now the length of each polymer $\ell_1, \ldots, \ell_r$, $\Sigma \ell_i = \ell'$ we expect the corresponding number of configurations to have the scaling form

$$\omega_{r\ell_1 \ldots \ell_r}(|i-j|) \sim \omega_{r\ell'}(|i-j|) \ell'^{-(r-1)} G\left(\frac{\ell_1}{\ell'} \ldots \frac{\ell_r}{\ell'}\right). \tag{27b}$$

Taking $\ell_1 = \ldots = \ell_r = \ell$ and integrating with respect to $j$ gives then the number of configurations of $r$ polymers each of length $\ell$ which relate fixed neighbourhoods of two different points

$$\omega_{r\ell} \sim \mu^{r\ell} \ell^{(2-2x_r)\nu-r} \tag{28a}$$

i.e. using (18) and (22)

$$\omega_{r\ell} \sim \mu^{r\ell} \ell^{(52-32r-9r^2)/32}. \tag{28b}$$

We have defined these numbers using fixed neighbourhoods of points $i$ and $j$ because we worked on a lattice and it is not possible in this case to put polymers starting from one point with no other intersection if $r$ is large enough. Naturally the exponent of (28) does not depend on the precise way the polymers behave in the neighbourhood of $i$ and $j$ and we can reformulate (28) by saying that $\omega_{r\ell}$ gives the number of configurations of $r$ polymers of length $\ell$ which are attached by their extremities. It would be interesting to evaluate these $\omega_{r\ell}$ by continuum models (des Cloizeaux and Jannink 1986, Duplantier 1986).

## 3. The percolation problem

The bond percolation problem can be obtained by the analytic continuation to $q=1$ of the $q$-state Potts model (see Wu 1982 and references therein). If the Hamiltonian of the magnetic model is

$$\beta\mathcal{H} = -\beta \sum_{\langle ij \rangle} (\delta_{\sigma_i\sigma_j} - 1) - H \sum_i \delta_{\sigma_i 1} \tag{29}$$

where the $\sigma_i$ are Potts variables ranging from 1 to $q$, the partition function in the limit $q \to 1$ becomes

$$Z(\beta, H) \simeq \sum_{\mathcal{G}} (1-e^{-\beta})^{B^o} (e^{-\beta})^{B^e} \prod_{\mathcal{C}} (q-1+e^{H})^{S_{\mathcal{C}}}. \tag{30}$$

In this formula the sum is taken over bond percolation graphs $\mathcal{G}$ with $B^o$ occupied bonds and $B^e$ empty bonds. The product is over all the clusters of $\mathcal{G}$ (including isolated sites) and $S_{\mathcal{C}}$ is the number of sites in the cluster $\mathcal{C}$.

The partition function (30) corresponds to a bond percolation problem with occupancy probability $p = 1 - e^{-\beta}$. In the limit $q \to 1$ the model (29) is thus critical at $\beta^c = -\log(1-p^c)$ where $p^c$ is the percolation threshold.

In zero magnetic field, the partition function (30) is simply

$$Z = 2^B \tag{31}$$

where $B$ is the total number of bonds. Since this result is in fact independent of the geometry and the temperature, we deduce in a way similar to (6) and (7) that the central charge $C$ is zero for the percolation problem, a result which agrees with the conjecture of Dotsenko and Fateev (1984) giving $C(q)$ for the model (29) in the case for $q = 1$. For calculating (30) one can use the transfer matrix method already introduced in Blöte *et al* (1981), Blöte and Nightingale (1982) and Nightingale and Blöte (1983). In the case $H = 0$, the matrix breaks into two submatrices, a matrix $M$ acting on configurations with at least one site connected to the 'ghost site' and a matrix $M'$ acting on configurations with no site connected to the ghost site.

Blöte and Nightingale (1982) and Nightingale and Blöte (1983) have shown that $M$ describes correlations of spin type while $M'$ describes correlations of energy type. The first dimensions which are deduced from the eigenvalues of these two matrices at momentum $K = 0$ are given in tables 10 and 12 while the corresponding *a posteriori* estimates are given in 11 and 13. The lattice is the square lattice for which the threshold is known exactly $p^c = \frac{1}{2}$.

**Table 10.** Dimensions deduced from the first eigenvalues of the matrix $M$ at $K = 0$.

| $l$ | | | | |
|---|---|---|---|---|
| 2 | 0.1109 | 0.773 | | |
| 3 | 0.1077 | 1.197 | 1.590 | 1.590 |
| 4 | 0.1060 | 1.315 | 1.768 | 1.768 |
| 5 | 0.1052 | 1.390 | 1.867 | 2.504 |
| 6 | 0.1048 | 1.440 | 1.928 | 2.721 |
| 7 | 0.1046 | 1.474 | 1.967 | 2.880 |
| 8 | 0.1045 | 1.498 | 1.995 | 2.997 |

The results of the first column of tables 10 and 11 are in agreement with $X_{H_1} = \frac{1}{2}\eta = \frac{5}{48}$ (Nienhuis *et al* 1980). This dimension has already been studied by Derrida and de Sèze (1982) and Derrida and Stauffer (1985). In a similar way the results of the first column of tables 12 and 13 agree with $X_{T_1} = 2 - 1/\nu = \frac{5}{4}$ (den Nijs 1979). This confirms in an independent way the validity of hyperscaling for percolation which has been recently discussed by Jug (1985).

**Table 11.** *A posteriori* estimates corresponding to table 10. The identification of the dimensions is explained in the text.

| $l$ | | | | |
|---|---|---|---|---|
| 3 | 0.0991 | 1.421 | | |
| 4 | 0.1031 | 1.756 | 2.164 | |
| 5 | 0.1043 | 1.684 | 2.131 | 4.394 |
| 6 | 0.1042 | 1.649 | 2.118 | 3.913 |
| 7 | 0.1042 | 1.635 | 2.112 | 3.756 |
| Identified dimensions | $X_{H_1}=\frac{5}{48}$ $=0.1042$ | $X_{H_2}=\frac{77}{48}$ $=1.6042$ | $X_{H_1}+2$ $=2.1042$ | $X_{H_2}+2$ $=3.6042$ |

**Table 12.** Dimensions deduced from the first eigenvalues of $M'$ at $K=0$.

| $l$ | | | | |
|---|---|---|---|---|
| 3 | 1.026 | 2.284 | | |
| 4 | 1.010 | 2.110 | 2.648 | 3.135 |
| 5 | 1.142 | 2.388 | 2.810 | 3.420 |
| 6 | 1.169 | 2.575 | 3.082 | 3.602 |
| 7 | 1.187 | 2.709 | 3.219 | 3.723 |
| 8 | 1.199 | 2.806 | 3.325 | 3.807 |
| 9 | 1.208 | 2.879 | 3.409 | 3.865 |

**Table 13.** *A posteriori* estimates corresponding to table 12. The identification of the dimensions is explained in the text.

| $l$ | | | | |
|---|---|---|---|---|
| 4 | 1.288 | | | |
| 5 | 1.269 | 3.600 | 4.697 | 4.323 |
| 6 | 1.261 | 3.477 | 4.420 | 4.219 |
| 7 | 1.256 | 3.388 | 4.270 | 4.159 |
| 8 | 1.254 | 3.303 | 4.161 | 4.113 |
| Identified dimensions | $X_{T_1}=\frac{5}{4}=1.25$ | $X_{T_1}+2=3.25$ | $X_{T_2}=4$ | 4 |

The magnetic series for the percolation problem has been conjectured by Dotsenko and Fateev (1984) to be

$$X_{H_l}=2h_{3/2,t+1/2}=(g(2t-1)^2-4)/48. \tag{32}$$

The second dimension of table 11 is in good agreement with the value $X_{H_2}$ deduced from (32). In table 11 one can also verify the presence of $X_{H_1}+2$ and $X_{H_2}+2$. Since the matrix does not break into several submatrices as was the case for polymers it is difficult to identify the next dimensions (32) in the spectrum and we do not present them here.

The thermal series of Dotsenko and Fateev (1984) is a line of the conformal grid in table 6

$$X_{T_l}=2h_{1,t+1}=[(3t+1)^2-1]/12. \tag{33}$$

In table 13 we identify, beside $X_{T_1}+2$, two dimensions in agreement with the value $d=4$. One of them corresponds to $X_{T_2}$ (33) and the other is the dimension of $L_{-2}\bar{L}_{-2}I$ (Cardy 1985) which was also observed for the self-avoiding walk problem. As was the case for the magnetic series, it is difficult to identify the following terms of the series (33) in the transfer matrix spectrum.

Although we have just identified the beginning of the series (32) and (33) we think that our results confirm the conjectures (32) and (33) of Dotsenko and Fateev (1984) for the $q$-state Potts model in the case $q=1$. Our results also confirm the general picture of the percolation transition which has been recently discussed by Jug (1985). We should finally mention that we have obtained very similar results for the site percolation problem.

We can turn now to the question of the corrections to scaling in percolation. The values of $\Delta$ which have been found by different numerical methods range from 1.2 to 1.8 (Adler *et al* 1983 and references therein).

Before the dimension already found by Nienhuis (1982b), $X_{T_2}=4$, which corresponds to $\Delta=\frac{8}{3}=2.67$, the only dimension we have found in the transfer matrix spectrum is $X_{T_1}+2=\frac{13}{4}$. This gives a value $\Delta'=\frac{5}{3}=1.67$ in good agreement with numerical estimates. Unfortunately for the same reasons as for the self-avoiding walk problem, we do not expect the corresponding operator to contribute to corrections to scaling. It is not clear to us whether we have not found all the operators or whether the experimental values of $\Delta$ can be explained by numerical difficulties or non-linear effects in the corrections.

## 4. Conclusion

We have shown how the numerical study of the transfer matrix spectrum at the critical point can give some information on models which are not completely solved by the conformal invariance theory. In this work we have mainly studied the thermal and magnetic series for the polymer and percolation problems. A curious feature is that these two models have the same central charge $C=0$. Moreover, although they are believed to belong to different universality classes, they possess the same infinite series of magnetic exponents while their thermal series are given by two different lines of the conformal grid of table 6. It would be interesting to see whether there is a precise geometrical correspondence between these two problems which could explain these analogies.

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## References

Adler J 1983 *J. Phys. A: Math. Gen.* **16** L515

Adler J, Moshe M and Privman V 1983 *Percolation Structures and Processes* (Ann. Israel Phys. Soc.) vol 5, ed G Deutscher, R Zallen and J Adler (Bristol: Adam Hilger) p 397

Adler J and Privman V 1982 *J. Phys. A: Math. Gen* **15** L417

Aharony A and Fisher M 1983 *Phys. Rev.* B **27** 4384
Andrews G E, Baxter R J and Forrester P J 1984 *J. Stat. Phys.* **35** 193
Belavin A A, Polyakov M A and Zamolodchikov B A 1984 *Nucl. Phys.* B **241** 333
Blöte H and Nightingale M P 1982 *Physica* **112A** 405
Blöte H, Nightingale M P and Cardy J L 1986 *Phys. Rev. Lett.* **56** 742
Blöte H, Nightingale M P and Derrida B 1981 *J. Phys. A: Math. Gen.* **14** L45
Cardy J L 1984 *J. Phys. A: Math. Gen.* **17** L385
—— 1985 *Preprint*
—— 1986 Private communication
de Gennes P G 1972 *Phys. Lett.* **38A** 339
den Nijs M P M 1979 *J. Phys. A: Math. Gen.* **12** 1857
Derrida B 1981 *J. Phys. A: Math. Gen.* **14** L5
Derrida B and de Sèze L 1982 *J. Physique* **43** 475
Derrida B and Stauffer D 1985 *J. Physique* **46** 1623
des Cloizeaux J and Jannink G 1986 *Polymères en solution, leurs modélisations et leurs structures*, (Paris: Editions de Physique) to be published
Dotsenko VL S 1984 *Nucl. Phys.* B **235** 54
Dotsenko VL S and Fateev V A 1984 *Nucl. Phys.* B **240** 312
Djordjevic Z V, Majid I, Stanley H E and dos Santos R J 1983 *J. Phys. A: Math. Gen.* **16** L519
Duplantier B 1986 *Phys. Rev. Lett.* **57** 941
Enting I G 1980 *J. Phys. A: Math. Gen.* **13** 3713
Enting I G and Guttman A J 1985 *J. Phys. A: Math. Gen.* **18** 1007
Friedan D, Qiu Z G and Schenker S 1984 *Phys. Rev. Lett.* **52** 1575
Gujrati P D 1981 *Phys. Rev.* A **24** 2096
Huse D A 1984 *Phys. Rev.* B **30** 3908
Jug G 1985 *Phys. Rev. Lett.* **55** 1343
Kac V G 1979 *Group thoretical methods in physics* ed W Beiglbock and A Bohm (*Lectures Notes in Physics* **94**) (Berlin: Springer) p 441
Klein D J 1980 *J. Stat. Phys.* **23** 561
Nienhuus B, Riedel E K and Schick M 1980 *J. Phys. A: Math. Gen.* **13** L18
Nienhuis B 1982a *Phys. Rev. Lett.* **49** 1062
—— 1982b *J. Phys. A: Math. Gen.* **15** 199
Nightingale M P and Blöte H 1983 *J. Phys. A: Math. Gen.* **16** L657
Polyakov M A 1970 *JETP Lett.* **12** 381
Privman V 1984 *Physica* **123A** 428
Privman V and Fisher M 1983 *J. Phys. A: Math. Gen.* **16** L295
Rapaport D C 1985 *J. Phys. A: Math. Gen.* **18** L201
Saleur H and Derrida B 1986 *J. Stat. Phys.* **44** 225
Sarma G 1978 *Ill Condensed Matter, Les Houches 78* ed R Balian, G Toulouse and R Maynard (Amsterdam: North-Holland) p 537
Wegner F 1972 *Phys. Rev.* B **5** 4529
—— 1976 *Phase Transitions And Critical Phenomena* vol 6, ed C Domb and M S Green (New York; Academic) p 8
Wu F Y 1982 *Rev. Mod. Phys.* **54** 235